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(Almost) Gibbsian Description of the Sign Fields of SOS Fields

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An example is presented of a measure on a lattice system which has a measure zero set of points (configurations) where some conditional probability can be discontinuous, but does not become a Gibbs measure under decimation (or other) transformations. We also discuss some related issues.

KEY WORDS: Non-Gibbsian measures; weak versus strong non-Gibbsianness; robustly non-Gibbsian measures; projected Gibbs measures; cluster models.

1. INTRODUCTION

In recent years (see refs. 52, 53, 51, 48, 49, 33, 55, 29, 43, 7, 45, 39, 40, 44, 54, 27 and references therein), various measures on finite-spin lattice models have been found which are not Gibbs measures in a strict sense. These measures can occur in physically rather natural set-ups, for example by applying single Renormalization Group maps to Gibbs measures (see also the fundamental work of refs. 24, 25, 28), as scaling limits of such maps, as stationary measures under some non-equilibrium evolution, or by directly constructing them.

As for many applications the availability of an "effective Hamiltonian" is desirable, or even essential, and also because the theory of Gibbs measures (a good description of which is in ref. 19) has proved to be such a versatile and fruitful tool in various domains, beside statistical mechanics where it originated (ergodic theory and dynamical systems, pattern recognition, (Euclidean) field theory...), a more general version of Gibbsian theory seems worth looking for.

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There have been essentially two approaches in which some of the non-Gibbsian counterexamples might be brought back into the Gibbsian fold. One, which was first suggested by R. L. Dobrushin,[8] and which was further discussed and studied in refs. 37, 13, 40, 41, 54, 6, 44, 35, 38 considers a measure on a finite-spin lattice system to be *weakly* non-Gibbsian—or what we would call *almost Gibbsian*—if the set of boundary conditions at which some conditional probability is essentially discontinuous (this set is empty for Gibbs measures) is non-empty but has measure zero; or, in a presumably weaker version, the set of boundary conditions on which the—to be constructed—interaction diverges has measure zero. This approach is suggested by an analogy with infinite-range unbounded-spin systems. If the set of such "bad" boundary conditions has positive measure, we will say that the measure is *strongly* non-Gibbsian. (As a side remark we mention that another class of models where one has Gibbs measures defined on a "large" set of well-behaved configurations but "exotic" behavior on its complement, are Ising—thus bounded-spin—models for spin-glasses, with very long-range (square summable but not absolutely summable) random pair interactions at high temperatures.[17, 18, 56, 57])

In the other approach, which is due to F. Martinelli and E. Olivieri,[42, 43] one studies what happens if one applies sufficiently many decimation transformations to a non-Gibbsian measure. If the measure remains non-Gibbsian, we will call it a *robustly* non-Gibbsian measure, otherwise (in case the transformed measure becomes Gibbsian) the non-Gibbsianness will be said to be *non-robust*.

There are examples known (for example the decimated Ising model at low temperature in a weak field) in which the non-Gibbsianness is both weak and non-robust, and also examples (J. van den Berg, and C. Maes–K. Vande Velde, private communication and ref. 36) in which one has a one-dependent measure in which the non-Gibbsianness is strong but non-robust (after decimation the measure is a product measure, which is trivially a Gibbs measure).

Here we present some examples having the opposite property, that is, the measure, due to its anomalous large deviation properties, does not become Gibbsian under decimation (or more general transformations), thus it is robustly non-Gibbsian, but it has a measure zero set of discontinuity points, hence at the same time the non-Gibbsianness is both robust and weak.

Comment. As for the Renormalization Group origin and meaning of many of these non-Gibbsian results, we suspect that in some of these examples it may in fact be the case that any description in terms of finite-spin variables runs into some kind of trouble, and it is not clear to us at present that the "almost Gibbsian" framework will be the most appropriate one to implement Renormalization-Group arguments in. Thus the conclusion (Bricmont's title of his contribution to the January 1997 Palaiseau meeting on Gibbsian versus non-Gibbsian measures, describing the results of ref. 6) that "Renormalization-group pathologies are not so bad" seems too early to draw, if it is founded only on the possibility of a description of some of the non-Gibbsian measures in terms of almost surely defined interactions. In other words, the renormalization-group pathologies which were found[24, 25, 28, 53] in trying to renormalize (Ising) spin-Hamiltonians may indicate that the proper objects to renormalize are different objects altogether (probabilities, contour Hamiltonians, or whatever the case may be, depending on the physical problem at hand), even if the Gibbsian formalism also has a wider validity than the usual treatments suggest.

2. THE EXAMPLE: THE SIGNS OF THE SOS-MODEL

We consider the (three-valued) sign-field of a 2-dimensional SOS-model at low temperatures. Similar arguments apply at suffiently low temperatures in higher dimensions (where one is always below the roughening transition). Our arguments apply also to the case where one considers a discrete Gaussian instead of an SOS-model.

Let the SOS-configuration space be $\{Z\}^{Z^2}$. Consider the Gibbs measure μ_0 for the Hamiltonian

$$H = \sum_{\langle i, j \rangle} |S_i - S_j| \tag{1}$$

at sufficiently large inverse temperature β, corresponding to the zero boundary conditions. Then the interface is centered at height zero (as we are below the roughening transition, our measure exists).

We define $\sigma_i = \text{sign}\, S_i$, where sign $0 = 0$. We now have the following result:

Theorem 1. The marginal measure μ^1 of the σ-variables, defined by μ_0, is an example of a measure whose non-Gibbsianness is both weak and robust.

Proof. First we show the (at worst) *weak* non-Gibbsian property, i.e., the measure μ^1 has conditional probabilities which for some version are continuous on a set of configurations of full measure.

A configuration is a point of continuity—in the product topology—if, considered as a boundary condition, changing it sufficiently far away does not influence the distribution of the variables in a fixed finite volume by much (sometimes this is referred to as quasilocality, for finite-spin systems continuity and quasilocality are equivalent). In our example the influence from boundary conditions can be completely shielded off for a set of configurations of measure one. Namely, we observe that in the S-variables there is in the Peierls regime[3,4] an ocean of zeroes, with islands in it, for typical (μ_0-almost all) configurations. As the zeroes in the S-variables and in the σ-variables coincide, any finite volume will, also with μ^1-probability one, be enclosed by a *-connected contour of zeroes. Changing the configuration outside this contour has no influence whatsoever on what occurs inside the contour. (In other words, instead of having, as for Gibbs measures,[47] surely the almost Markov property, here we have almost surely the—local—Markov property.)

The second thing we need to prove is that the measure μ^1 is *robustly* non-Gibbsian. To do this, we want to show that it has anomalous large deviation properties, in the sense that the probability of finding all σ-spins to be plus in some volume decays slower than exponentially with the size of the volume. For this we apply a variation of some essentially known arguments. First we observe that shifting the spins in, say, a square volume Λ of size N by N by a height difference n costs an energy $n \times |4N|$. Moreover, conditioned on this shifted average, the fluctuations in the square around this shifted height n, with high probability are smaller than $O(\ln N)$ ([4], Appendix 3, part 2). By choosing for example $n = O(N^{\varepsilon})$, we find that the probability of finding all σ-spins to be plus decays not faster than $\exp\{-N^{1+\varepsilon}\}$.

A weaker version of a similar result has been obtained by J. Lörinczi ([34] and private communication).

By similar arguments as in refs. 30, 11, and 53, Section 4.4, it then follows that μ^1 cannot be a Gibbs measure; neither can it become so under either deterministic[43] or stochastic[54] renormalization group transformations.

3. THE INTERACTION FOR THE SIGN FIELD

Once we know that at μ_0-almost all configurations all the conditional probabilities are continuous, we can try and write them in a Gibbsian form for an (almost everywhere convergent) interaction, similarly to what was done by Dobrushin, Maes–Vande Velde, and Bricmont–Kupiainen–Lefévere in different examples.

As a side remark we note that one might have an almost surely convergent interaction and at the same time nowhere continuous conditional probabilities; in fact this is the typical situation for infinite-range unbounded-spin systems where changing the boundary condition to a bad one (a rapidly increasing one) far away, might strongly influence the distribution at the origin, whatever happens in intermediate regions.

In the cases mentioned above, to get an almost everywhere convergent (exponentially decaying) interaction or relative energy was rather complicated, as one needs to control cluster expansions which cannot be uniformly convergent (that is, uniformly in each configuration of the system), a problem which is similar to what happens in the Griffiths phase of disordered systems[23] (where the convergence is not uniform in the disorder configuration) or also the high-temperature phase of models with many-body interactions,[9, 50] and which was solved there by various methods (compare refs. 16, 1, 12, 5, 17, 20, 21, 22).

In our case this problem turns out to be much simpler, as for the above indicated set of configurations of μ_0-measure 1, we not only have continuity of the conditional probabilities, but we even have the Markov property. This allows us in a simple way to obtain a lattice gas (or vacuum) potential, with the all-zero configuration playing the role of the vacuum. Indeed, let us define the interaction to be just the energy of a particular island configuration. Here the island is any finite connected set. The energy is obtained by taking the logarithm of the ratio of the two probabilities: one is the probability of the finite island configuration of pluses and/or minuses (delimited by the ocean of zeroes from the outside and may be also by the lakes of zeroes from the inside), while the second one is the probability of the all-zero configuration on the island, both computed with zero boundary condition around the island. By definition, our interactions are nonzero only on such islands. As the configurations consisting of finite islands in a sea of zeroes have full measure (because we are in the Peierls regime) we are done.

The resulting interactions are somewhat reminiscent of those of the (one-dimensional) Fisher–Felderhof cluster models, the statistical mechanics of which has been found to have many unusual properties, and which are well-known to fall outside the usual Gibbsian (DLR) formalism (see for example refs. 14, 15). In the theory of interacting particle systems such cluster models belong to the reversible (but non-attractive) case of "nearest particle systems" (ref. 31, Chap. 7). It was observed that non-trivial invariant measures may live on a reduced configuration-space, because of the existence of one or a few (= countably many) dangerous configurations. Dangerous configurations are configurations containing one or two (semi-) infinite clusters. Our example could be viewed as a higher-dimensional generalization of such cluster models. (Our islands correspond to the clusters). One should note that a "particle" in a nearest particle system corresponds to a hole in a cluster model and a zero in our example; they give the values of the variables which can shield off the influence from the outside.

The nearest particle condition (which corresponds to the fact that clusters—islands—at distance two or more do not interact) is what we have called the almost surely holding Markov property.

An unusual property of such models is that they may have a weakly Gibbsian high temperature measure, but a degenerate measure at low or intermediate temperatures, namely a "frozen" one, which is supported on one or finitely many configurations. Related to this is the possible occurrence of "ideally metastable" frozen states,[46] which cannot happen in the ordinary Gibbsian formalism.

Connected to this is that, although in our example the existence of a nontrivial measure being concentrated on the finite-island configurations follows by construction, in general the question for which choices of finite-island-configuration probabilities there exist such a measure is probably not easy to decide.

4. A GENERAL EXAMPLE

The goal of this section is to present a further example, which shows that the passage from the Gibbs fields to the almost Gibbs fields brings with it some new properties, which show that the almost Gibbs fields have to be treated with more care than their usual counterparts. More specifically, we are going to argue that the relaxation of the property of continuity of the conditional probabilities to that of a.s. continuity of them is not so innocuous as one would like to think (the authors included) and leads to the appearance of some features not encountered in the realm of the proper Gibbs fields.

Motivated by the analysis of the field of signs of the SOS-model, we present in this section a general construction of a class of almost Gibbs fields, which are not classical Gibbs fields. We present here the simplest possible version of this general construction. As the reader will see, as soon as one does not have to ensure the convergence of the relative energy for all boundary conditions, all the constructions become quite simple. Our example will be a class of random fields $\sigma = \{\sigma_t\}$ on $\mathbb{Z}^d$, $d \geqslant 2$ with values 0 or 1, i.e. a lattice gas model. To define it, we need some preliminary notions.

A subset $C \subset \mathbb{Z}^d$ will be called connected, iff the subset of $\mathbb{R}^d$, formed by the union of closed unit cubes centered at the sites of C is connected. Two points x, $y \in \mathbb{Z}^d$ will be called neighboring, if the set $\{x, y\} \subset \mathbb{Z}^d$ is connected. A sequence of neighboring points will be called *a path*. A boundary ∂C of a set C is the set of all points in the complement C^c, which have at least one neighbour in C. A connected finite set will be called a *polymer*. A connected infinite set will be called an *infinite polymer*.

A polymer $C \subset \mathbb{Z}^d$ will be called a *c-polymer* iff the complement C^c is also connected. The boundary of a c-polymer is connected as well. If the complement C^c is disconnected, then the polymer C is "hollow", and will therefore be called an *h-polymer*. The support supp(C) of the polymer C is defined as the set $C \cup \partial C$. The core of the polymer B, which will be denoted by core (B), is by definition the smallest c-polymer, containing B. If for a finite set A we have $A = \text{supp}\,(C)$ for some c-polymer C, then it is easy to see that such a c-polymer C is uniquely defined by A.

There is an evident one-to-one correspondence between the subsets of the lattice and the configurations σ. In what follows we will use the notation $\mathscr{C}(\sigma)$ to denote the set of polymers of the configuration σ.

We remind the reader that the Gibbs potential $\mathscr{U}$ is a family of functions, $\mathscr{U} = \{U_A(\sigma_A), A \subset Z^d, |A| < \infty\}$, where $\sigma_A \in \{0, 1\}^A$ is any configuration on A. We are going to introduce what we will call *polymer potentials*: We say that the Gibbs potential $\mathscr{U}$ is a polymer potential, if for some constants k, K, $0 < 2k < K < \infty$ the following holds:

$$U_A(\sigma_A) = \begin{cases} u_C & \text{if } A = \text{supp}(C) \text{ for some c-polymer } C\text{, an d } \sigma_A = (\text{ind}_C)|_A, \\ v_C & \text{if } A = \text{supp}(C) \text{ for some h-polymer } C\text{, and } \sigma_A = (\text{ind}_C)|_A, \\ 0 & \text{otherwise,} \end{cases}$$

where the parameters u_C, v_C satisfy the bounds:

$$k|\partial C| \leqslant u_C \leqslant K\,|\partial C| \qquad \text{for the c-polymer } C$$

$$v_C \geqslant K\,|\text{core}(C)| \qquad \text{for the h-polymer } C$$

(Here ind_C is the indicator function of C.) In fact, we need to suppose more about the function u_C defined for all c-polymers C. To this end let ρ be some fixed integer, and V_ρ be a cubic box centered at the origin. Let $b(\cdot)$ be a nonnegative function on Ω_{V_ρ}, which vanishes exactly for two configurations: $\sigma \equiv 0$ and $\sigma \equiv 1$. We suppose that the function u_C admits the following representation:

$$u_C = \sum_{x \in \mathbb{Z}^d} b(\text{ind}_C(\cdot + x)|_{V_\rho}) \equiv |\partial C|_b \tag{2}$$

Clearly, such a function takes values which are of the order of the boundary $|\partial C|$ of the polymer C. But in addition to that, relation (2) ensures that this boundary term is a sum of local terms along ∂C. Note that the "length" $|\partial C|$ is itself a functional which admits such a representation.

If V is now any finite set in $\mathbb{Z}^d$, and $\sigma_V \in \Omega_V$ is any configuration on V, we define the energy $H^h(\sigma_V)$ by

$$H^h(\sigma_V) = \sum_{A \subset V} U_A(\sigma_V|_A) - h \sum_{t \in V} \sigma_t$$

The real parameter h is called the *magnetic field* (or chemical potential). If in addition a configuration σ_{V^c} is specified in the complement of V, which is called a *boundary condition*, we define the relative energy $H^h(\sigma_V \mid \sigma_{V^c})$ by

$$H^h(\sigma_V \mid \sigma_{V^c}) = \sum_{A \cap V \neq \varnothing} U_A((\sigma_V \cup \sigma_{V^c})|_A) - h \sum_{t \in V} \sigma_t$$

Note that the infinite clusters of particles, which might appear in the configuration $\sigma_V \cup \sigma_{V^c}$, do not contribute to the relative energy. The probability distribution $q_V^{\beta, h}$ on Ω_V, given by

$$q_V^{\beta, h}(\sigma_V \mid \sigma_{V^c}) = \frac{\exp\{-\beta H^h(\sigma_V \mid \sigma_{V^c})\}}{Z(V, \beta, h, \sigma_{V^c})}$$

is called *specification*, and the normalizing factor $Z(V, \beta, h, \sigma_{V^c})$ is the *partition function*. Note that the specification is a discontinuous function of σ_{V^c}: every boundary condition σ_{V^c} with an infinite cluster of particles is a point of discontinuity. In case of the boundary condition σ_{V^c}, which is identically zero, we will use the shorter notation $q_V^{\beta, h}(\sigma_V)$ for the specification, and will call the probability distribution $q_V^{\beta, h}(\sigma_V)$ the *Gibbs distribution in V with zero boundary condition.* The measures $q_V^{\beta, h}(\cdot)$, viewed as measures on the set $\Omega_{\mathbb{Z}^d}$, form a compact family, parametrized by the volumes V. Therefore for every value of the parameters β and h it has at least one limit point, as a weak limit of measures on a compact set. For this no continuity conditions are required. Let us choose such a limit point for every β, h and denote it by $\mu^{\beta, h}$.

Theorem 2. Suppose that the temperature is low enough, i.e. $\beta > \beta_0 \gg 1$. Then there exists a value $h_0 = h_0(\beta) > 0$ of the magnetic field, such that:

(1) For $0 \leqslant h \leqslant h_0$ the measures $\mu^{\beta, h}$ are nontrivial. That means that the probability $\mu^{\beta, h}(\sigma_V)$ of the appearance of a configuration σ_V in a finite box V is positive for any V and σ_V.

(2) For all $h > h_0$ the measures $\mu^{\beta, h}$ are concentrated on a single configuration $\sigma \equiv 1$.

Proof. We begin by introducing the ensemble of the *exterior cores.* A c-polymer D will be called an exterior core of the configuration $\sigma_V \in \Omega_V$ iff D is a maximal set inclusionwise, which can be written as $D = \bigcup_{\{C_i \in \mathscr{C}(\sigma_V) : \mathrm{core}(C_i) \cap \mathrm{core}(C_j) \neq \varnothing\}} \mathrm{core}(C_i)$. In particular, the *exterior core of the origin*, $D_0 = D_0(\sigma_V)$, is the union:

$$D_0(\sigma_V) = \bigcup_{\{C \in \mathscr{C}(\sigma_V) : 0 \in \mathrm{core}(C)\}} \mathrm{core}(C)$$

Clearly, there is a polymer C_0 of the configuration σ_V, such that

$$\mathrm{core}(C_0) = D_0(\sigma_V)$$

In what follows, we will use the notation $\mathscr{D} = \mathscr{D}(\sigma)$ to denote the set of all exterior cores of the configuration σ. Our first step will be to write down the distribution of the exterior cores, induced by the measures $q_V^{\beta, h}$.

Lemma 3. If β is large enough, then there exists a functional ϕ on the set of all exterior cores, such that for any $\sigma \in \Omega_V$

$$q_V^{\beta, h}(\sigma : \mathscr{D}(\sigma) = \mathscr{D}) = \Xi(V, \phi, h)^{-1} \prod_{D \in \mathscr{D}} \exp\{-\phi(D) + \beta h\, v(D)\} \tag{3}$$

The functional ϕ is a τ-functional, which means that

$$\phi(D) \geqslant \tau\, |\partial D|$$

with $\tau = \tau(\beta) \to \infty$ as $\beta \to \infty$, and $\Xi(V, \phi, h)$ is a partition function. Moreover, as a functional of the boundary of the core, ϕ is almost local, and the following representation holds:

$$\phi(D) = \beta\, |\partial D|_b + g^\beta(D)$$

The function g has the following properties:

$$\text{for any x, we have } g^\beta(D) = g^\beta(D + x);$$

$$|g^\beta(D)| \leqslant \exp\{-k\beta\, |D|\}.$$

Proof. First we will define the functional ϕ. Let D be a finite c-polymer, $\Gamma = \partial D$ be its boundary, and V be a box containing D. Consider the partition function

$$Z(D) \equiv Z(V, D, \beta, h, \sigma_{V^c} \equiv 0) = \sum_{\sigma_V \in \Omega(\{D\})} \exp\{-\beta H^h(\sigma_V \mid \sigma_{V^c})\}$$

which is calculated over all configurations σ which have D as their only exterior core. Define $\phi(D) \equiv \phi(\Gamma)$ by

$$\phi(\Gamma) = -\ln Z(D) + \beta h\,|D|$$

The reason for our lemma to hold lies in the simple structure of the energy function; the main contribution to the above partition function comes from the configuration $\sigma_V \in \Omega(\{D\})$ which is identically $+1$ inside D. We have:

$$\begin{aligned}
Z(D) &= \exp\{-\beta(u_D - h\,|D|)\} \\
&\quad + \sum_{\substack{C:\, C \text{ is an } h\text{-polymer} \\ \text{core}(C) = D}} \exp\{-\beta(v_C - h\,|C|)\} \\
&= \exp\{\beta h\,|D|\} \Bigg[\exp\{-\beta u_D\} \\
&\quad + \sum_{\substack{C:\, C \text{ is an } h\text{-polymer} \\ \text{core}(C) = D}} \exp\{-\beta(v_C + h(|D| - |C|))\} \Bigg]
\end{aligned}$$

Denote the last sum by $Z^{\text{hollow}}(D)$. It can be bounded from above by

$$\begin{aligned}
Z^{\text{hollow}}(D) &= \sum_{\substack{C:\, C \text{ is an } h\text{-polymer} \\ \text{core}(C) = D}} \exp\{-\beta(v_C + h(|D| - |C|))\} \\
&\leqslant 2^{|D|} \exp\{-2\beta k\,|D|\}
\end{aligned}$$

which completes the proof. ∎

Once we have the representation (3), the proof of Theorem 2 will be complete provided we will demonstrate that in the contour model (3) the following holds: for $h \leqslant h_0$ the typical configuration consists of small external contours, while for $h > h_0$ it contains one large external contour which occupies essentially the whole box V. This is enough, since, as the next lemma shows, the following holds: given the polymer C_0, the probability of seeing a zero inside core(C_0) goes to zero as the polymer C_0 gets bigger.

Lemma 4. For all $h \geqslant 0$

$$q_V^{\beta, h}(\sigma_t = 0 \mid D_0(\sigma_V) = D) \leqslant \exp\{-\beta c(h)\,|D|\}$$

where the function $c(h) > 0$, uniformly in $h \geqslant 0$.

Proof. The proof is easy, due to the very simple structure of the potential. First of all, the conditional distribution $q_V^{\beta,h}(\cdot \mid D_0(\sigma_V) = D)$ coincides with the distribution $q_D^{\beta,h}(\cdot \mid D_0(\sigma_D) = D)$. Consider the configuration $\sigma_D^+ \equiv 1$. Then

$$H_D^h(\sigma_D^+) \leqslant K\,|\partial D| - h\,|D|$$

For $\sigma \in \Omega_D$ different from σ_D and satisfying the condition $D_0(\sigma_D) = D$ we have

$$H_D^h(\sigma) \geqslant k\,|D| - h\,|D|$$

Since the number of configurations in Ω_D is $2^{|D|}$, we have:

$$q_D^{\beta,h}(\sigma_D \neq \sigma_D^+ \mid D_0(\sigma_D) = D) \leqslant \exp\{-(k\beta - \ln 2)\,|D| + K\,|D|\} \quad \blacksquare$$

The last thing left is the study of the contour ensemble (3). In the special case of the function b, when the values $|\Gamma|_b$ are equal to the length of the contour Γ, this study is a subject of ref. 38. The general case can be treated exactly in the same manner. The main idea is to replace the ensemble of external contours (3) by the usual contour model of Pirogov–Sinai theory (maybe with a positive parameter). That amounts to solving the system of equations

$$\begin{aligned}\exp\{[\beta h - f(\psi)]^+ v(\Gamma)\}\,\exp\{-\psi(\Gamma)\}\,Z(\mathrm{Int}(\Gamma) \mid \psi)\\ = \exp\{-\phi(\Gamma) + \beta h v(\Gamma)\}\end{aligned} \tag{4}$$

where $[a]^+ = \max\{a, 0\}$. Here $\psi = \psi_{\beta,h}$ is the unknown contour functional, $f(\psi)$ is its free energy, and $Z(\mathrm{Int}(\gamma) \mid \psi)$ is the usual partition function of the contour model. In addition to the proof of the existence of the solution of (4) we need to know two properties of it:

1. ψ is a τ-functional, that is $\psi(\Gamma) \geqslant \tau\,|\Gamma|$, with $\tau \to \infty$ as $\beta \to \infty$;

2. ψ is almost local, in the following sense: there exists a function $G(x, \Gamma)$, defined for all contours Γ and all sites $x \in \partial(\mathrm{Int}(\Gamma))$, such that G is translation invariant and

$$|G(x, \Gamma_1) - G(x, \Gamma_2)| \leqslant \exp\{-c\,\mathrm{dist}(t, \Gamma_1\,\Delta\Gamma_2)\}$$

with $c \to \infty$ as $\beta \to \infty$, while

$$\psi(\Gamma) = \sum_{x \in \partial(\mathrm{Int}(\Gamma))} G(x, \Gamma) + a\,v(\Gamma)$$

with $a \geqslant 0$.

(Under the second property one is able to show that it is very unlikely that a contour develops a long and thin protuberance—the statement, which is the part of our theorem. In general this is not the case.) The constructions of [38] can be carried on in the present situation, and they enable one to prove all the above statements. In particular, the value $h_0(\beta)$ is given by the relation:

$$h_0(\beta) = \inf\{h : \beta h \geqslant f(\psi)\} \tag{5}$$

For the values of $h > h_0(\beta)$ our system is described by the contour model with a positive parameter (see (4)), and this is the reason why the typical configuration in this regime contains one large polymer of the size of the system. The fact that the value $h_0(\beta)$ is strictly positive might appear a bit surprising; but the equation (5) clearly implies this positivity, since the free energy $f(\psi_{\beta, h})$ is positive even for $h = 0$. We notice that in the case of a small field the weak convergence of the measures actually implies the almost Gibbs property with the set of configurations without infinite polymers being the good set. Indeed, we need to consider for this only finite-polymer probabilities whose existence is guaranteed by the weak convergence of measures in the thermodynamic limit, and whose non-triviality is demonstrated in our Theorem 2. In the case of a strong magnetic field, the limit measure has been shown above to be a point measure.

5. COMMENTS AND CONCLUSIONS

The properties of the examples we have presented, as well as those of the Fisher–Felderhof cluster models we mentioned in Section 3, imply that, even if for a weakly Gibbsian measure one can define an almost everywhere defined interaction, these interactions allow for far more singular behavior than is allowed in the standard theory. In particular, the strict and strong convexity properties of the pressure[10, 26] do not hold; as a function of the magnetic field (or the chemical potential) the pressure is linear on either a positive or negative half-line. This follows by similar arguments as are given in [11] for the projection on 2-valued Ising spins of massless continuous Gaussians (compare [30] and [53]). For these projected continuous Gaussians at present we do not know if they are weakly or strongly non-Gibbsian, although their non-Gibbsian character is robust.[43, 54] These projected continuous Gaussians occur in two rather different contexts, in the study of entropic repulsion,[30] as well as scaling limits for majority type transformations in high dimensions.[11]

The Fisher–Felderhof models can be seen as one-dimensional almost-surely finite-range models, but this almost-sure as opposed to a uniform condition on the interactions allows for much wilder behavior (phase-transitions, "anti-phase-transitions" and so on, in one-dimensional short-range models), and in Renormalization Group language it does not seem plausible that one should try to classify them in one and the same universality class.

Thus, even if one can define such almost surely defined, not uniformly convergent interactions, what their existence implies, and what applications they can be used for, is not clear at present.

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